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The variances in a single-mode squeezed vacuum state

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Abstract : In this research paper, we calculate the variances in single-mode pure and mixed-squeezed vacuum states. Direct relationships between the variances and the mean number of photons, present in the coherent state of the mode, are obtained. The results assure that the squeezed states have less uncertainty in one quadrature than the coherent state. Some interesting properties of the squeezed states are followed as consequences of introducing the P-representation to describe the state of the mode.

Keywords : The squeezed states, the variances, quadrature components, electric field.

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1. Introduction

At time t and frequence ω, the electric field for a nearly monochromatic plane wave may be decomposed into two quadrature components with time dependence cos ωt and sin ωt. The present paper describes the squared uncertainties (variances) of these components in a single-mode squeezed vacuum pure and mixed states.

It is found that when the density operator for a pure squeezed vacuum state represents the mode, the fluctuations in both the quadratures are still to vary with the squeeze factor, which varies from zero to ∞. On the other hand, when the mode is represented by the P-representation [1–4] of the density operator for squeezed vacuum state, the fluctuations in two quadratures are dependent only on the mean number of photons present in the coherent state of this mode.

In these two cases, it is found also that the squeezed states have less fluctuation in the other quadrature than the zero-point fluctuation at the expense of increased fluctuation in the other quadrature phase. The zero-point fluctuation represents the standard quantum limit to the reduction of noise in a signal.

The total noise in each of pure and mixed squeezed vacuum states is estimated. Actually, the total noise in a state is a more fundamental quantity than the total number of photons. It is conserved if the total number of photons is conserved, but the converse is not true [5]. However, our results assert that there are direct relationships between the total noise in a state and the number of photons, including the occupation of the ground state of the mode, present in this state.

In the case of the mixed squeezed vacuum state, an upper limit for the product of the deviations in two quadratures is established. Of course, the lower limit of this product can be fixed as well by means of Heisenberg's uncertainty relation.

2. Definitions

This section is composed of two subsections. Subsection 2.1 defines the single-mode squeezed states. Subsection 2.2 defines the dependence of the mean occupation number on the mean free path.

2.1. *Single-mode squeezed states :*

The single-mode squeeze operator [6–10] is defined in general, by

$$S(z) = \exp\left|\frac{1}{2}z^*a^2 - \frac{1}{2}z(a^+)^2\right.$$

$$0 \le r \le \infty, \frac{\pi}{2}, -<\varphi \le \frac{\pi}{2}, \quad (1)$$

where the complex factor z takes the form

$z = r\exp(2i\varphi)$. (2)

The real numbers r and φ are called, respectively, the squeeze factor and squeeze angle of a squeezed state, while a and a^+ are the annihilation and creation operators. Since each one of a and a^+ does not commute with the operator $\exp(-2i\varphi)a^2 - \exp(2i\varphi)(a^+)^2$, it is easy to show that

$$\begin{aligned} S^+(z)aS(z) &= a\cosh r - a^+\exp(2i\varphi)\sinh r, \\ S^+(z)a^+S(z) &= a^+\cosh r - a\exp(2i\varphi)\sinh r. \end{aligned} \quad (3)$$

These relations signify that the single-mode squeeze operator (1) mixes the operators a and a^+. Therefore, it induces a correlation between the position and momentum variables.

The singl-mode squeezed vacuum pure state, symbolized by $|z\rangle$, is defined as operator $S(z)$ acting on the ground state $|0\rangle$ of the mode [5,10], namely,

$|z\rangle = S(z)|0\rangle$. (4)

2.2. Dependence of the mean number of photons on the mean free path :

Let Ω be a unit vector coinciding with the direction of the velocity of a particle when its energy is u. We introduce the function $g(\mu_0, u - u')$ to represent the relative probability of the particle being left with parameters (Ω', u) as a result of a collision before which the pair characterized it was (Ω', u'). The number $\mu_0 = \Omega'. \Omega'$ is the cosine of the angle through which the particle is scattered. Let us assume that $g(\mu_0, u - u')$ may be expanded in terms of the Legendre polynomials $P_j(\mu_0)$ such as

$$g(\mu_0, u-u') = \frac{1}{4\pi}\sum_{j=0}^{\infty}(2j+1)g_j(u-u')P_j(\mu_0), \quad (5)$$

with

$$g_j(u-u') = \int_{-1}^{+1} d\mu_0 g(\mu_0, u-u')P_j(\mu_0), \quad (6)$$

where we have used the normalization conditions of Legendre polynomials [11]. Let us now define the numbers α and β as follows

$$\alpha = \int_{u-\gamma}^{u}(u-u')g_0(u-u')du', \quad (7)$$

$$\beta = \int_{u-\gamma}^{u} g_1(u-u')du'. \quad (8)$$

In which, the functions $g_0(u - u')$ and $g_1(u - u')$ are resulted from (6) for j = 0, 1 respectively, while γ represents the maximum energy loss (the maximum energy loss occurs when the particle is scattered through angle of 180°). Accordingly, the average numbers of photons symbolized by ε, occupied by the coherent state of the mode as a function of the mean free path $T(u)$ of the atoms of a medium through which the natural light passes, takes the form [12–15] :

$$\varepsilon = \frac{1}{3\alpha(1-\beta)}\int_0^u du'T^2(u'). \quad (9)$$

Actually, owing to relation (9), one may measure, experimentally, the average number of photons occupied by a coherent state of field mode. That is by measuring the mean free path T. In this paper, explicit relations between the variances in a single-mode squeezed vacuum mixed-state and the number ε will be constructed.

In spite of the simplicity of relation (9) which can be obtained as a result of applying a very crude model of interaction between light and atoms, it is an essential parameter in the quantum-mechanical theory of light.

3. The density operator for a single-mode squeezed vacuum state

If we have a complete knowledge about the state of a physical system, one can say with certainty that every element of this system is in that state. For such system, the density operator can be defined simply as the dyadic product [1] of that state. Thus, if we are sure that the system we study is in the squeezed vacuum pure state $|z\rangle$ defined by (4), then the density operator describing that system takes simply, the form [16–18] :

$$\tilde{\rho} = |z\rangle\langle z|. \quad (10)$$

Operator (10) is called the density operator for a single-mode squeezed vacuum pure state. If we insert state (4) and its complex conjugate into the right side of (10), then we can easily deduce that

$$\mathrm{Tr}\,\tilde{\rho} = 1. \quad (11)$$

In the other hand, if we do not have complete knowledge about the state of the field mode we are studying, and we are not sure that it is in state (4) (say), then we can assume that the probability that it is in this state is $Q(z)$. In this case, the density operator symbolized by ρ, is the superposition of the projection operator (10), [16–18], *i.e.*

$$\rho = \int\int Q(z)|z\rangle\langle z|d^2z\,. \tag{12}$$

This state is in the form of the P representation [1,4] for the density operator. It is called the single-mode sqeezed vacuum mixed-state. According to notation (2), we can easily show that the differential element $d^2z(=d(\mathrm{Re}\,z)d(\mathrm{Im}\,z))$ takes the form

$$d^2z = 2rdrd\varphi. \tag{13}$$

As a density operator, ρ must be Hermitian and its trace is unity. namely,

$$\mathrm{Tr}\,(\rho) = 1. \tag{14}$$

In fact, since the single-mode squeezed vacuum pure state $|z\rangle = S(z)|0\rangle$ (where $S(z)$ is the unitary operator (1)) is normalized to unity, the constraint (14) provides that

$$\int\int Q(z)\,d^2z = 1\,. \tag{15}$$

In terms of representation (2) of z and the number ε defined by (9), the weighting factor $Q(z)$ assumes the form [12–15] :

$$Q(z) = (\pi\varepsilon^3)^{-1/2}\,r\cos\varphi\exp(-r^2/\varepsilon);$$

$$0 \le r < \infty, -\frac{\pi}{2} < \varphi \le \frac{\pi}{2}. \tag{16}$$

It is clear that function (16) is not positive everywhere in the plane, but it is positive for all values of r, on the segment $-\pi/2<\varphi \le \pi/2$, only. Thus, this function cannot be considered as a probability density, but it is a new quasi-probability distribution function. This function distributes in phase angle through its dependence on $\cos\varphi$. Actually, if we insert the element of area (13) as well as the function (16) into the left side of (15), then we can easily show that

$$\int\int Q(z)d^2z = 2(\pi\varepsilon^2)^{-1/2}$$

$$\times\int_{\pi/2}^{\pi/2} d\varphi\cos\varphi\int_0^{\infty} drr^2\exp(-r^2/\varepsilon) = 1. \tag{17}$$

That assures the fulfilment of the constraint (15).

4. The variance in squeezed states

This section is composed of three subsections. The first defines the quadrature components of the electric field. The second and the third consider the variance in these components relative to the squeezed vacuum pure and mixed states.

4.1. Quadrature components of the electric field :

The electric field for a nearly monochromatic plane wave with frequency ω, may be written at the time t as, [10] :

$$E(t) = \lambda[a\exp(-i\omega t) + a^+\exp(i\omega t)]\,, \tag{18}$$

where λ is a constant including the spatial wave function. The amplitude a can be written in the following complex form :

$$a = \xi + i\varsigma, \tag{19}$$

with ξ and ς are Hermitian operators, which obey the commutation relation

$$[\,\xi,\varsigma\,] = i/4. \tag{20}$$

Now in terms of ξ and ς, the field amplitude (18) takes the form :

$$E(t) = 2\lambda(\xi\ \cos\ \omega t + \xi\ \sin\ \omega t). \tag{21}$$

Accordingly, ξ and ς may be identified as the amplitudes of two quadrature phases of the field.

By means of the commutation relation (20), we can deduce the following relation for the uncertainties $\Delta\xi\left(=[V(\xi)]^{1/2}\right)$ and $\Delta\zeta\left(=[V(\varsigma)]^{1/2}\right)$ ($V(x)$ denotes the variance in x) in ξ and ς

$$\Delta\xi\Delta\varsigma \ge 1/4\,. \tag{22}$$

In fact, the coherent states have $\Delta\xi\Delta\varsigma \ge 1/4$ [1,10], while the squeezed states [19–22] have, as we shall see later, $\Delta\xi\Delta\varsigma \ge 1/4$. Let us consider now the complex rotated transformation

$$q + ip = (\xi + i\varsigma)\exp(-i\varphi)\,, \tag{23}$$

with φ is the squeeze angle. Solving relation (23) with respect to q and p, to have

$$q = \frac{1}{2}\left[a\exp(-i\varphi) + a^+\exp(i\varphi)\right], \tag{24}$$

$$p = -\frac{i}{2}\left[a\exp(-i\varphi) - a^+\exp(i\varphi)\right], \tag{25}$$

where relation (19) as well as its complex cojugate have been used.

Knowing now that the variances in a quantum state are defined as the mean-square deviations or fluctuations of the real and imaginary parts of the annihilation operator a, [5,10]. These mean-square fluctuations should be evaluated, of course, with respect to this state. The sum of these variances gives the total noise in this state. Accordingly, the variances in the state of the mode we consider, are the squared uncertainties in ξ and ς, defined by (19). But the mean-square fluctuations in ξ and ς are, respectively, the same as those in q and p. Operators ξ and ς related to q and p through (23) which differs from (19) only by the phase factor exp $(-i\varphi)$.

4.2. Variances in q and p relative to state (10) :

Relative to operator (10), the variance in q defined by (24), is given by

$$\tilde{V}(q) = \mathrm{Tr}(\tilde{\rho} q^2) - \left[\mathrm{Tr}(\tilde{\rho} q)\right]^2. \tag{26}$$

If we insert operator (10) and (24) into the right side of (26), then we can evaluate $\tilde{V}(q)$ through two successive steps. In the first, we use state vector (4) and then we apply the unitary property of $S(z)$. In the second step, we use the operator relation (3), and then we apply the well known identities : $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^+|n\rangle = \sqrt{n+1}|n\rangle$, $a|0\rangle = 0$ as well as orthonormalization condition of Fock's states $|n\rangle$. Accordingly, we obtain in this case, for the variances in q, the expression

$$\tilde{V}(q)\frac{1}{4}\exp(-2r). \tag{27}$$

Again, if we follow the same technique, we can show that the variance in p defined by (25), with respect to state (10) is in the form :

$$\tilde{V}(p) = \frac{1}{4}\exp(2r). \tag{28}$$

In fact, the variance in squeezed state $|\mu, z\rangle$ (μ denotes the associated coherent state) have equally, the forms (27) and (28), which are the variances in the squeezed vacuum state $|z\rangle$ (see [10]). This means that in the case of the squeezed pure states, the associated coherent states do not contribute to the variances in these states. It is clear from (27) and (28) that the squeezed vacuum state (10) has less fluctuation in the quadrature q than a coherent state at the expense of increased fluctuations in p.

The sum of the variances (27) and (28) gives the total noise in state (10) (see [5]). Thus, if $\tilde{N}$ denotes the total noise of the pure state (11), then we can easily show that

$$\tilde{N} = \frac{1}{2}\cosh 2r. \tag{29}$$

Indeed, since the state of the mode is considered as a pure state, the mean-square fluctuations of q and p in addition to the total noise $\tilde{N}$, are still functions in the squeeze factor r, which varies from zero to ∞. In the case of the squeezed vacuum mixed-state, these quantities depend only on the mean occupation number of photons present in the coherent state of the mode. As well, from (27) and (28), we can easily deduce the following relation for the uncertainties $\Delta q\left(=\left[\tilde{V}(q)\right]^{1/2}\right)$ and $\Delta p\left(=\left[\tilde{V}(p)\right]^{1/2}\right)$ in q and p :

$$\Delta q \Delta p = 1/4. \tag{30}$$

which is still, as in coherent state, the minimum value allowed by the quantum mechanical theory.

4.3. Variances in q and p relative to state (12) :

Now, if the field mode is represented by the squeezed vacuum mixed-state (12), then the mean-square deviation in q defined by (24), may be written, similarly, as

$$V(q) = \mathrm{Tr}(\rho p^2) - \left[\mathrm{Tr}(\rho q)\right]^2$$

$$\int Q(z)\left\langle z\left|q^2\right|z\right\rangle d^2z - \left[\int Q(z)\left\langle z|q|z\right\rangle d^2z\right]^2. \tag{31}$$

Actually, it is easy to show that $\langle z|q|z\rangle = 0$, while $\langle z|q|z\rangle = \frac{1}{4}\exp(-2r)$. Accordingly, (31) becomes

$$V(q) = \frac{1}{4}\int\int Q(z)\exp(-2r)d^2z \tag{32}$$

By means of constraint (15), the right side of (32) is the expectation value of (1/4) exp$(-2r)$, which is the variance in q relative to the pure state (10). Inserting function (16) and the element of area (13) into the right side of (32), to have

$$V(q) = \frac{1}{2(\pi\varepsilon^2)^{1/2}}\int\int_{\pi/2}^{\pi/2}\cos\varphi d\varphi$$

$$\times \int^{\infty} r^2 \exp\left[-(r^2 + 2\varepsilon r)/\varepsilon\right]dr. \tag{33}$$

In fact, the integrals in the right side of (33) can be easily carried out to obtain

$$V(q) = \frac{1}{2(\pi\varepsilon)^{1/2}}\left[(1+2\varepsilon)\upsilon(\varepsilon) - \varepsilon\right], \tag{34}$$

where the function $\upsilon(e)$ is defined by

$$\upsilon(\varepsilon) = \int_0^\infty \exp\left[-(r^2 + 2\varepsilon r)/\varepsilon\right] dr. \tag{35}$$

Indeed, by the same technique, we can prove that the mean-square fluctuation in p defined by (25), when the field mode represented by density operator (12), takes the form

$$V(p) = \left(\varepsilon + \frac{1}{2}\right)\exp(\varepsilon) - \frac{1}{2(\pi\varepsilon)^{1/2}}\left[(1 + 2\varepsilon)\upsilon(\varepsilon) - \varepsilon\right]. \tag{36}$$

The total noise in state (12) is then determined by the sum of the variance (39) and (41). This sum, symbolized by N, is equal to

$$N = \left|\ \varepsilon + \frac{1}{2}\ \right| \exp(\varepsilon). \tag{37}$$

In contrast with the pure state, here the mean-square fluctuations of q and p as well as the total noise N, are independent of the squeezed factor r. These variance are dependent exclusively on the number ε. Physically, this number is the mean number of photons occupied by the coherent state of the mode. Formally, ε is given integral (9), which involves the mean free path T of medium's atoms through which the thermal light passes.

In fact, we can by a little mathematical manipulation, prove that $\nu(\varepsilon) \sim 1/2$. According to this asymptotic expansion, the variances (34) and (36) reduce to

$$V(q) = 1/4\sqrt{\pi\varepsilon}\ . \tag{38}$$

$$V(q) = (\varepsilon + 1/2)\exp(\varepsilon) - 1/4\sqrt{\pi\varepsilon}\ . \tag{39}$$

Relations (38) and (39) give the variance in q and p relative to the single-mode squeezed vacuum state (12). These relations depend on the number ε, which is defined by expression (9). In fact, it is clear that the squeezed states have $V(q) \neq V(p)$. A family of minimum uncertainty states is defined by taking the equal sign. One of such class of minimum uncertainty states is that of the coherent states, which have $V(q) = V(p) = 1/4$. Once again, relation (38) and (39) indicate that the squeezed vacuum mixed-states, as in the case of squeezed vacuum pure states, have less fluctuation in q than the coherent state at the expense of increased fluctuation in p.

Now, from (38), it is clear that

$$V(q) < \frac{1}{4}, \tag{40}$$

this inequality is considered, by Walls [10], as the condition for squeezing.

5. Concluding remarks

By means of the identity : (1/2) $\cosh 2r = \sinh^2 \pm r + 1/2$, we can write (29), which gives the total noise in the squeezed vacuum pure state (10), as

$$\tilde{N} = \sinh^2 r + 1/2. \tag{41}$$

On the other hand, the total noise in the squeezed vacuum mixed-state (12) is given by (37), which can be rewritten in the form :

$$N = \left|\left|\ \varepsilon + \frac{1}{2}\ \right|\exp(\varepsilon) - \frac{1}{2}\ + \frac{1}{2}. \tag{42}$$

In fact, the first term in the right side of (41) and the quantity between the square bracket in the right side of (42) give, respectively, the mean number of photons present in states (10) and (12), see [10] and [14]. Each of the rest terms in the right sides of (41) and (42) is of the value 1/2, that represents the contribution of the ground state of the mode to the total noise. Hence, we conclude that the total noise in a state is equal to the mean number of photons, including the contribution of the ground state occupied by this mode. Thus the total noise of a state can be thought of as the noise content of the state in units of photon number, including a half quantum due to the zero point noise. This fact has been mentioned by Schmaker [5].

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